LEVEL #

BR66435

UNLIMITED

ROYAL AIRCRAFT ESTABLISHMENT

①

DDC
RECEIVED
APR 10 1979
C

THE IMPLEMENTATION AND PRACTICAL VERIFICATION OF A SUPERPOSITION METHOD

FOR THE SOLUTION OF ELASTIC CRACK PROBLEMS

by

P. Bartholomew

November 1978

ROYAL AIRCRAFT ESTABLISHMENT

(6)

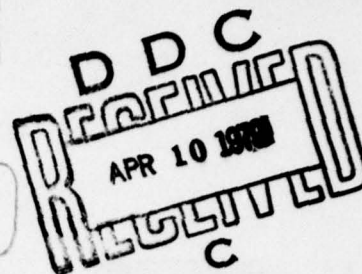## THE IMPLEMENTATION AND PRACTICAL VERIFICATION OF A SUPERPOSITION METHOD FOR THE SOLUTION OF ELASTIC CRACK PROBLEMS,

by

(18) DRIC   (10) P. Bartholomew   (12) 21 p.

(19) BR-66435

SUMMARY

This Memorandum demonstrates a superposition method which provides a practical and efficient method for the solution of linear elastic crack problems.

The method approximates the linear elasticity solution over a substructure containing the crack tip by the superposition of appropriate singular stress fields and a relatively coarse finite element mesh. A hybrid variational principle is employed to ensure displacement compatibility between this substructure and adjoining ones employing finite elements alone.

A key feature of the method is that it may be implemented by adding routines to existing finite element packages. The particular implementation discussed here is intended for use in the aircraft industry.

2

LIST OF CONTENTS

1   INTRODUCTION

   Linear theory predicts singular stresses at crack tips[1] which create serious consequences when solutions are sought by numerical methods such as finite element analysis. It is known[2] that the energy convergence of conventional finite elements on refining the mesh is limited by the order of the singularity, and that it is not improved by the use of higher order elements. As a consequence, although there is still a limited amount of new work employing conventional analyses[3,4], much of the research effort in this area has been aimed at incorporating the singular stresses directly into the finite element formulation. This is usually done for a single element embedding the crack tip[5,6] or possibly for a group of elements adjacent to the crack tip[7,8]. The resulting super-elements are capable of modelling the stresses at the crack tip with considerable accuracy, while conventional elements are used to idealize the remaining structure. However to achieve overall accuracy further conditions must hold. Firstly there must be adequate displacement compatibility between the super-element and the surrounding conventional elements. Secondly those surrounding elements must themselves be small enough to give a close representation of the stresses adjacent to the super-element. This situation can lead to some conflict in the choice of dimensions for the super-element. It should be large enough to cover the region of large stress gradients dominated by the $1/\sqrt{r}$ term, yet small enough to interface with a sufficiently fine exterior mesh. If it is necessary to refine the exterior mesh the super-element is reduced in size, leaving still larger stress gradients to be idealized by the conventional elements. Thus the energy convergence using the super-element is no better than that achieved by the conventional finite elements[2] in the presence of the singularity.

   This Memorandum discusses a method which restores the full rate of convergence to the finite element scheme by enabling the singular functions to be used over a fixed region about the crack tip, while interfacing with an arbitrarily fine conventional finite element mesh on the exterior. The interior region, which is treated here as a substructure, represents stress and displacement fields by the superposition of finite elements and singular trial functions. The finite elements are only required to model discrepancies between the exact solution and the singular trial function, and are most significant at the boundary at the substructure.

   Although the method gives correct convergence with mesh refinement, it is also important that it be competitive with super-elements for the coarsest meshes if it is to be of value in an engineering context.

The theoretical development of the method is described in Ref 9. Here we consider a formulation of the method which is suitable for use with commercial finite element packages. In particular the discussion relates to an RAE program currently installed on both the RAE and British Aerospace ICL computers.

Also sufficient numerical results are presented to enable the technique to be evaluated as a practical tool for handling singularities in linear elasticity. First, however, the method itself is outlined.

2   METHOD

The underlying idea of this work is that the accuracy of finite element analysis in the presence of stress singularities may be restored by the inclusion of singular trial functions appropriate to the problem. In this instance the functions correspond to stresses at a crack in a remotely loaded infinite sheet. It would be possible to include these extra trial functions in sufficient numbers to make the finite elements redundant. Here, however, the policy is to retain the flexibility of the finite element method by using the minimum numbers of singular fields, and further restricting their use to a finite region about the crack tip.

A variational principle is used to minimise the difference between the exact solution and the approximation. In this instance the superposition of a singular trial function and a mesh of simple displacement finite elements are used to form the approximation. There is some difficulty in maintaining displacement compatibility between these augmented trial functions and the regular finite elements of adjacent substructures. This is overcome by using a hybrid variational principle[10], and seeking the stationary point of the functional

$$\Pi_C = -U(\sigma^I) + \int_{\partial V} T^I \tilde{u} ds - \int_{C_T} \bar{T} \tilde{u} ds \ , \tag{1}$$

where $U(\sigma^I)$ is the total strain energy summed over all elements. σ^I and T^I are respectively stress fields and the corresponding tractions used as trial functions in the interior of each element, $\tilde{u}$ are displacement trial functions defined on the boundary ∂V of each element, and $\bar{T}$ are tractions prescribed on C_T .

The finite element used is the constant strain triangle whose piecewise linear displacements are denoted u^F . The continuous functions used to augment the solution are denoted u_i^* where the suffix i identifies the particular

function. In order to facilitate the enforcement of displacement continuity across the interface C_S between the local substructure and adjacent ones it is also convenient to introduce the finite element field $u_i^\dagger$ which takes the same values as the corresponding augmenting field u_i^* at the element nodes. The displacements defined on the perimeter of each element are then taken to be of the form

$$
\hat{u} = \begin{cases} u^F + \sum_i \alpha_i (u_i^* - u_i^\dagger) & \text{in the special region and on } C_T \\ u^F & \text{on } C_S \\ u^F = \bar{u} & \text{(prescribed on } C_K) \end{cases}
\tag{2}
$$

The unknowns represent the nodal connection quantities and the coefficients α_i associated with each augmenting trial function. This choice ensures that the additional trial functions take zero values at every node, enabling the use of nodal connection quantities q^F for enforcing compatibility between adjacent substructures in the usual manner.

The interior stress fields used are constructed by summing the constant stresses over each element and the singular fields $\alpha_i \sigma_i^*$ corresponding to the displacement fields $\alpha_i u_i^*$. Variation of (1) with respect to the trial stress fields yields the exact elasticity solution corresponding to the prescribed boundary displacements whenever the element perimeter does not coincide with C_S or C_K. Thus the constant fields σ^F and $\sigma_i^\dagger$ are introduced explicitly and correspond to the boundary displacements u^F and $u^\dagger$ respectively. The form used for the interior stress fields is then given by

$$
\sigma^I = \sigma^F + \sum_i \alpha_i \left(\sigma_i^* - \sigma_i^\dagger + \sigma_i^C \right) .
\tag{3}
$$

The correction terms σ_i^C are required only for the elements adjacent to the interface C_S or kinematic boundaries C_K, where the augmenting stress fields do not correspond to the piecewise linear displacements enforced on the boundary.

Substitution of terms (2) (3) into the functional (1) gives

Str 940

$$\Pi_C = U(u^F) - \int_{C_T} u^F \bar{T} ds$$

$$+ \sum_i \alpha_i \Bigg\{ - 2U(u^F, u_i^\dagger) + \sum_j \alpha_j U(u_i^\dagger, u_j^\dagger) + \int_{C_T} u^F T_i^* ds - \int_{C_T} \left(u_i^* - u_i^\dagger \right) \bar{T} ds$$

$$+ \tfrac{1}{2} \sum_j \alpha_j \int_{C_T} \left(u_i^* - 2u_i^\dagger \right) T_j^* ds + \int_{C_S + C_K} u^F T_i^* ds - \int_{C_S + C_K} \left(u_i^* - u_i^\dagger \right) T^F ds$$

$$+ \sum_j \alpha_j \Bigg[- \tfrac{1}{2} \int_{C_S + C_K} u_i^* T_j^* ds + \int_{C_S + C_K} \left(u_i^* - u_i^\dagger \right) T_j^\dagger ds - U\left(\sigma_i^C, \sigma_j^C \right)$$

$$- \int_{C_S + C_K} \left(u_i^* - u_i^\dagger \; T_j^C ds \right) \Bigg] \Bigg\} . \qquad (4a\text{-}m)$$

Values for the correction terms σ_i^C may be derived algebraically on an element by element basis. Variation of (4) with respect to σ_i^C gives

$$\sigma_{ni}^C = - \frac{1}{A} \frac{E}{(1 - \nu^2)} \int_1^2 \left(u_{ni}^* - u_{ni}^\dagger \right) ds$$

$$\sigma_{si}^C = - \frac{1}{A} \frac{E\nu}{(1 - \nu^2)} \int_1^2 \left(u_{ni}^* - u_{ni}^\dagger \right) ds \qquad \left.\right\} \qquad (5)$$

$$\tau_{nsi}^C = - \frac{1}{A} \frac{E}{2(1 + \nu)} \int_1^2 \left(u_{si}^* - u_{si}^\dagger \right) ds$$

where $u_i^C = \left(u_{ni}^C, u_{si}^C \right)$,

$$\sigma_i^C = \left(\sigma_{ni}^C, \sigma_{si}^C, \tau_{nsi}^C \right)$$

and $A = \tfrac{1}{2}(\ell_1 + \ell_2)h$ is the area of the element shown in Fig 1. If two sides of the element coincide with the boundary $C_S + C_K$ contributions from each

side are summed to give σ_i^C . Having determined these unknowns, the functional (4) may be reduced to a quadratic polynomial in α_i and q^F . Minimisation may then be achieved by the solution of a set of linear equations in the normal finite element manner.

3 IMPLEMENTATION

3.1 General framework

The implementation described generates a suitable substructure for inclusion in commercial finite element packages. Its stiffness matrix is presented in the partitioned form shown in Fig 2. The submatrices K_{II}, K_{BB}, K_{IB}, K_{BI} form the usual stiffness matrix assembled from constant strain elements, while P_I, P_B are the corresponding load vectors. The main package would then condense out the freedoms specified as being internal giving the usual reduced matrices

$$\bar{K}_{BB} = K_{BB} - K_{BI}K_{II}^{-1}K_{IB}$$

$$\bar{P}_B = P_B - K_{BI}K_{II}^{-1}P_I \quad . \tag{6}$$

The present method however supplies extra rows and columns K_{AI}, K_{AB}, K_{AA} of the stiffness matrix and terms P_A in the load vector corresponding to the additional unknowns α_i . For computational convenience these unknowns are treated as the freedoms associated with an extra node. Although these last freedoms will be local to the particular substructure it may nonetheless be convenient to treat them as boundary freedoms associated with the highest level of substructuring. In this way physically meaningful quantities such as the stress intensity factors may be determined without further processing of constituent substructures.

The following subsections show how the terms of functional (4) may be assembled into the augmented 'stiffness' matrix.

3.2 Contour integrals

Considering the terms of functional (4), the first two terms only are found in conventional finite element formulations, whilst several of the extra terms require integration over the boundary of the substructure.

Within the RAE program, the necessary values of u_i^* and T_i^* for the augmenting trial functions are supplied by a subroutine, and the integrations performed by six-point Gauss quadrature. The program accepts data corresponding to each segment of the boundary, defined in terms of its end nodes. The type of

Str 940

boundary condition is specified and the appropriate values for prescribed tractions $\bar{T}$ or displacements $\bar{u}$ are read. If no boundary conditions are defined at the perimeter of the substructure, a default option supplies interface boundary C_S conditions.

The terms

$$\int_{C_T} u^F T_i^* ds \tag{4e}$$

$$\int_{C_K + C_S} u^F T_i^* ds \tag{4h}$$

contribute to the submatrix (K_{AI}, K_{BI}). The symmetric part only of the terms

$$\int_{C_T} \left(u_i^* - 2u_i^\dagger \right) T_j^* ds \tag{4g}$$

$$\int_{C_S + C_K} u_i^* T_j^* ds \tag{4j}$$

is used to form the square submatrix K_{AA}. The augmenting part P_A of the load vector is formed from

$$\int_{C_T} \left(u_i^* - u_i^\dagger \right) \bar{T} ds \quad . \tag{4f}$$

Another integration performed at the same point in the program is

$$\int_{C_S + C_K} \left(u_i^* - u_i^\dagger \right) ds$$

which is required for the evaluation of the stresses σ_i^C from equation (5). The correction stresses σ_i^C are stored for later use.

The nodal loads corresponding to these stresses in each element also contribute to the submatrix (K_{IA}, K_{BA}) though the term

$$\int_{C_S + C_K} (u_i^* - u_i^\dagger) T^F ds \quad . \tag{4i}$$

The final integration

$$\int_{C_T} u^F \bar{T} ds \tag{4b}$$

to determine the usual finite element load vector (P_I, P_B) is simplified by the assumption that $\bar{T}$ is piecewise constant.

3.3 Area integrals

The simplicity of the elements reduces the area integrations to simple summation over the elements of the substructure. Expression of the first term in the form

$$U(u^F) = \frac{1}{2} q^F K q^F \tag{4a}$$

is a feature common to all finite element programs. Here the unaugmented stiffness matrix K is assembled from element stiffness matrices which have been calculated algebraicly rather than by numerical quadrature. The contributions to $[K_{IA}, K_{BA}]$ resulting from the terms $U(u^F, u^\dagger)$ are also evaluated on an element-by-element basis. The corresponding stresses $\sigma_i^\dagger$ are added to the storage containing σ_i^C and summation of the strain energy

$$U(\sigma_i^\dagger + \sigma_i^C, \sigma_j^\dagger + \sigma_j^C)$$

over the elements of the substructure gives the contributions to K_{AA} arising from the symmetric part of terms

$$U(\sigma_i^C, \sigma_j^C) \tag{4ℓ}$$

$$\int\limits_{C_S + C_K} \left(u_i^* - u_i^\dagger\right) T_j^C ds \;=\; 2U\left(\sigma_i^C, \sigma_j^C\right) \tag{4m}$$

$$\int\limits_{C_S + C_K} \left(u_i^* - u_i^\dagger\right) T_j^\dagger ds \;=\; 2U\left(\sigma_i^C, \sigma_j^\dagger\right) \tag{4k}$$

$$U\left(u_i^\dagger, u_j^\dagger\right) \tag{4d}$$

in (4).

This completes the assembly of the augmented substructure stiffness matrix. Additional organisational routines may be employed to rotate the reference frame from the local two-dimensional coordinates and embed it in the globally-defined three-dimensional system.

4   RESULTS

The validation of the superposition method in Ref 9 is performed using a single problem; namely a centre cracked square sheet under uniaxial end load. The points established are that convergence of the stress intensity factor with mesh refinement appears to be linear, and that greatest accuracy is obtained by employing the singular functions over intermediately sized regions. The method is also found to be particularly accurate when applied to such a problem. Two alternative approaches for determining stress intensity factors are employed. The first bases the estimate directly on the coefficients of the singular trial function, while the second uses the program to provide energy estimates which are then used to give stress intensity factors by strain energy release. The latter is marginally more accurate, but in each case accuracy of better than 2% is achieved for an absolutely regular mesh in which the element size is also the semi-crack length.

The test cases presented here are more stringent insofar as they include cracks growing from regions of high stress gradients or towards reinforcement. In the first instance the finite element mesh employed is only just adequate for the analysis of the uncracked structure; even so reasonably accurate estimates for the stress intensity factor are obtained.

These results for symmetric cracks growing from a circular hole are compared with those given by Rooke and Cartwright[11] in Fig 3. The points plotted are

obtained by strain energy release, based on analysis results using the finite element meshes shown with a/b $\approx$ 0.5, 0.6, 0.7, 0.8, 0.9 . Estimates obtained directly from the values of the coefficients α are in good agreement.

The same finite element idealizations are then used to determine the effect of various forms of loading applied to the hole, this providing a preliminary study of cracks growing from pin-loaded lugs. Results shown in Fig 4 corresponding to radial loading over the upper half of the hole are found to be in close agreement (within 2%) with those obtained by Parker[12] using collocation. The same figure shows that a point load gives a stress intensity factor up to 30% greater, the effect being most marked for shorter crack lengths. Fig 4 also shows that for these shorter crack lengths the effect of intermediate arc of contact between pin and lug more nearly resembles the point loaded case.

The particular lug of interest is shown in Fig 5. The stress intensity factors used by Kirkby and Rooke[13] for this problem are obtained by compounding[14] known solutions. Only the results for symmetric cracks are checked, but here agreement is sufficiently close to enhance confidence in each technique. The effects of rounding the end of the lug and of reducing its length are also considered.

The final problem applies the technique to a reinforced panel containing a series of in line cracks. A regular 12 × 12 mesh gave the points plotted in Fig 6 against comparison curves from Rooke and Cartwright[11] which agree to within 2%.

REFERENCES

No.	Author	Title, etc
1	M.L. Williams	Stress singularities resulting from various boundary conditions in angular corners of plates in extension. *J. Applied Mech.* 19, 526-528 (1952)
2	P. Tong T.H.H. Pian	On the convergence of the finite element method for problems with singularity. *Int. J. Solids Struct.* 9, 313-321 (1973)
3	D.M. Parks	A stiffness derivative finite element technique for determination of crack tip stress intensity factors. *Int. J. Fracture,* 10, 487-502 (1974)
4	Y. Murakami	A simple procedure for the accurate determination of stress intensity factors by the finite element method. *Eng. Fract. Mech.,* 8, 643-655 (1976)
5	E. Byskof	Calculation of stress intensity factors using the finite element method with cracked elements. *Int. J. Fract. Mech.,* 6, 159-168 (1970)
6	P. Tong T.H.H. Pian S.J. Lasry	A hybrid-element approach to crack problems in plane elasticity. *Int. J. Num. Meth. Eng.,* 7, 297-308 (1973)
7	D.M. Tracey	Finite elements for determination of crack tip stress intensity factors. *Eng. Fract. Mech.,* 3, 255-265 (1971)
8	S.N. Atluri A.S. Kobayashi M. Nakagaki	An assumed displacement hybrid finite element for fracture mechanics. *Int. J. Fracture,* 11, 257-271 (1975)
9	P. Bartholomew	Solution of elastic crack problems by superposition of finite elements and singular fields. RAE Technical Report 76165 (1976) *Computer Meth. in Appl. Mech. and Eng.,* 13, 59-78 (1978)
10	T.H.H. Pian P. Tong	Basis of finite element methods for solid continua. *Int. J. Num. Meth. Eng.,* 1, 2-28 (1969)
11	D.P. Rooke D.J. Cartwright	Compendium of stress intensity factors, HMSO (1976)

REFERENCES (concluded)

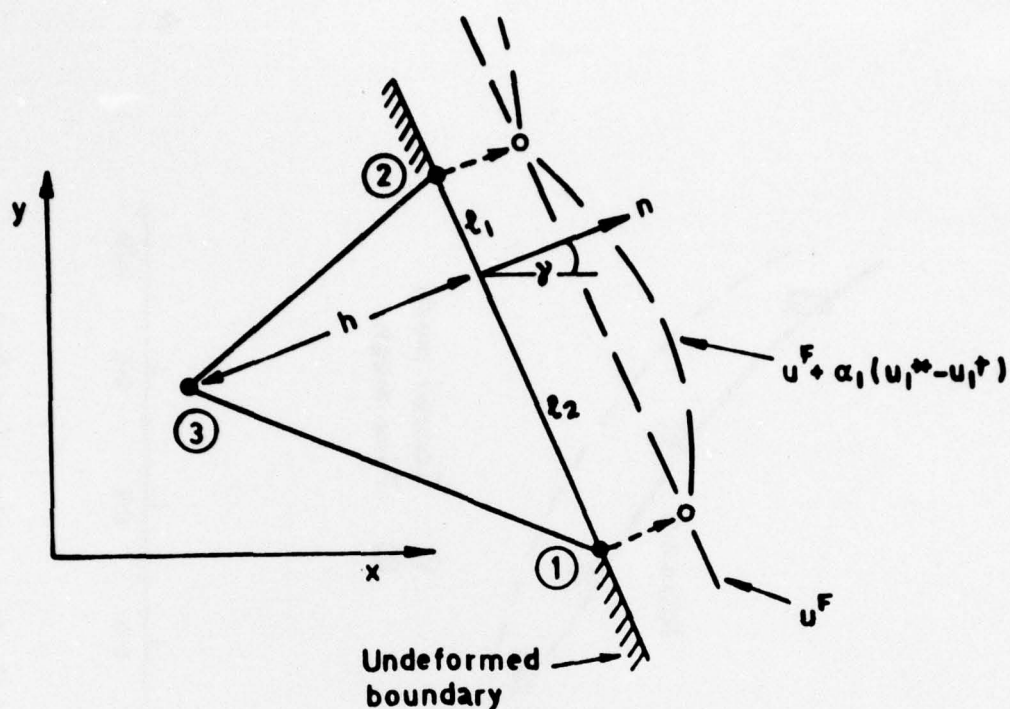| No.	Author	Title, etc
12	A.P. Parker	Private communication
13	W.T. Kirkby D.P. Rooke	Fracture mechanics in engineering practice. Applied Science Publishers, 339-360 (1977)
14	D.J. Cartwright D.P. Rooke	Approximate stress intensity factors compounded from known solutions. *Eng. Fract. Mech.*, 6, 563-571 (1974)

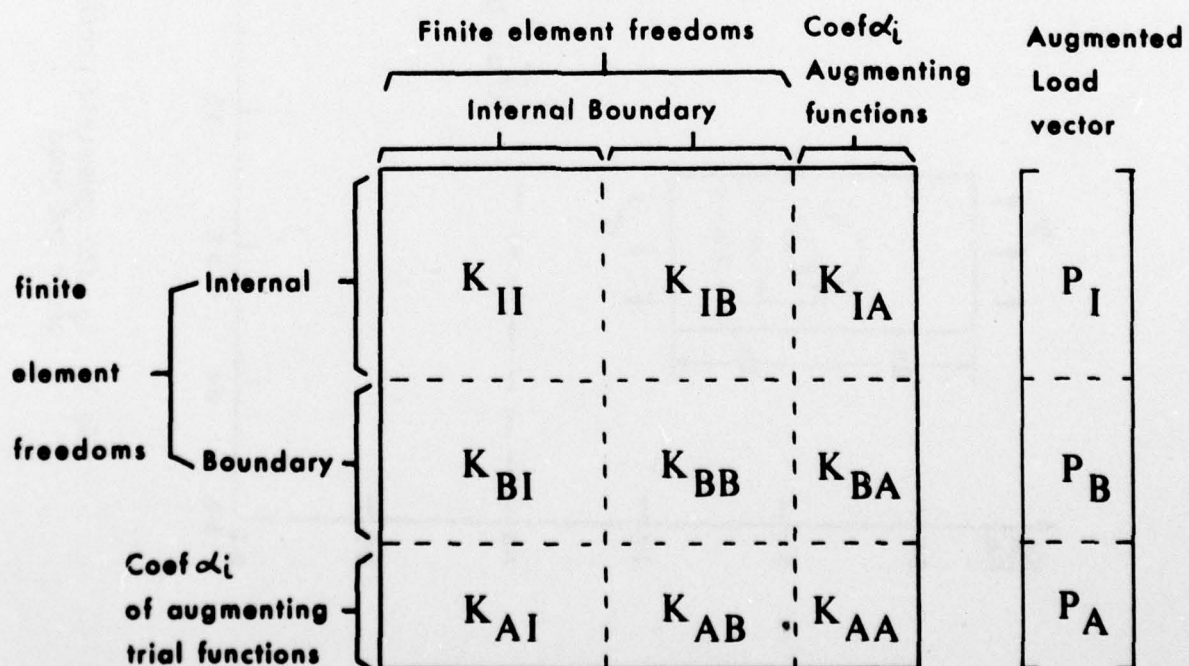Fig 1 Element geometry and boundary displacements
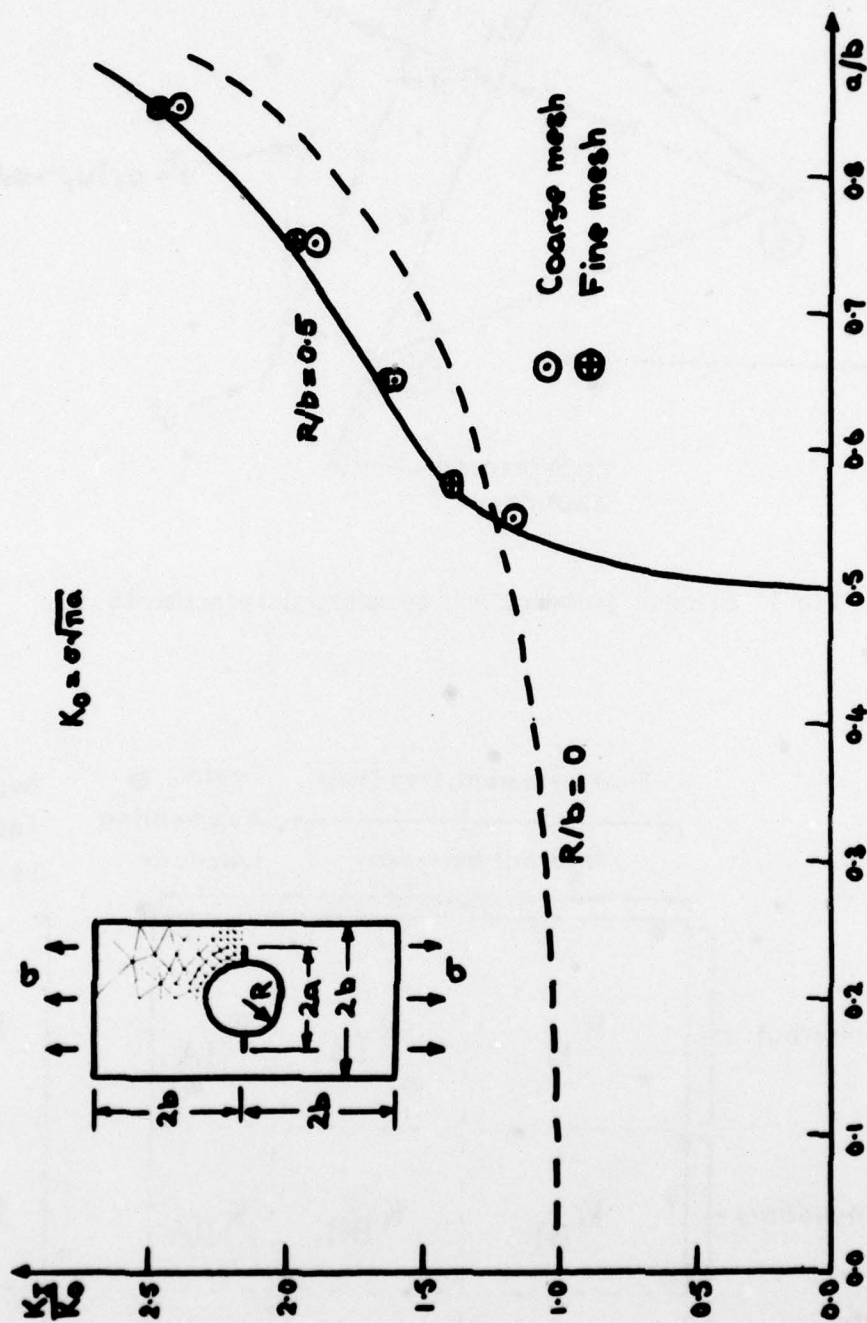

Fig 2 Partitioning of substructure stiffness matrix

Str 940

Fig 3

Fig 3 K_I for symmetric cracks at a circular hole in a rectangular sheet under end load
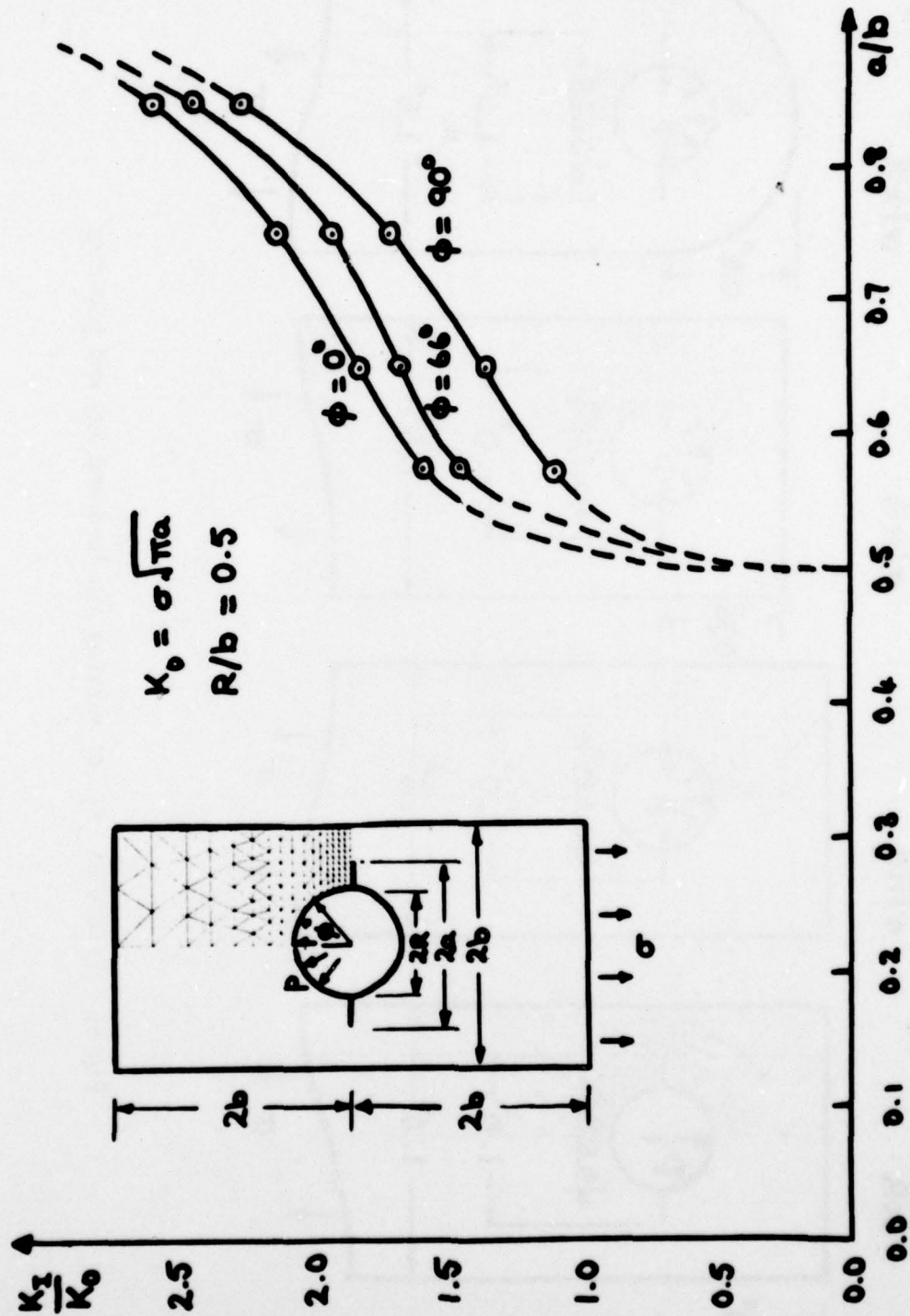
TM Str 940

Fig 4

$K_0 = \sigma\sqrt{\pi a}$

$R/b = 0.5$

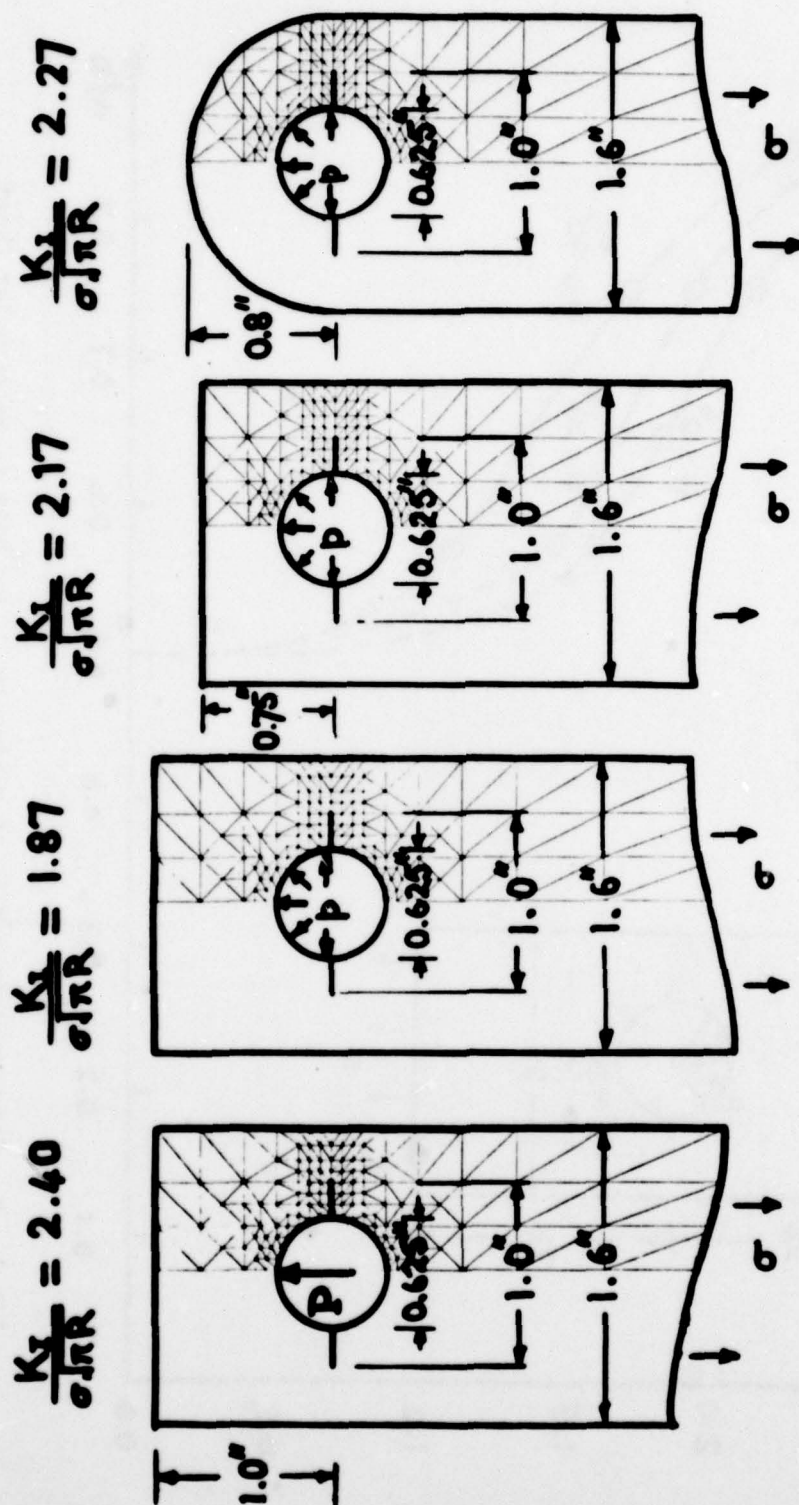Fig 4 K_I for symmetric cracks at variously loaded hole in rectangular sheet, reacted by end load

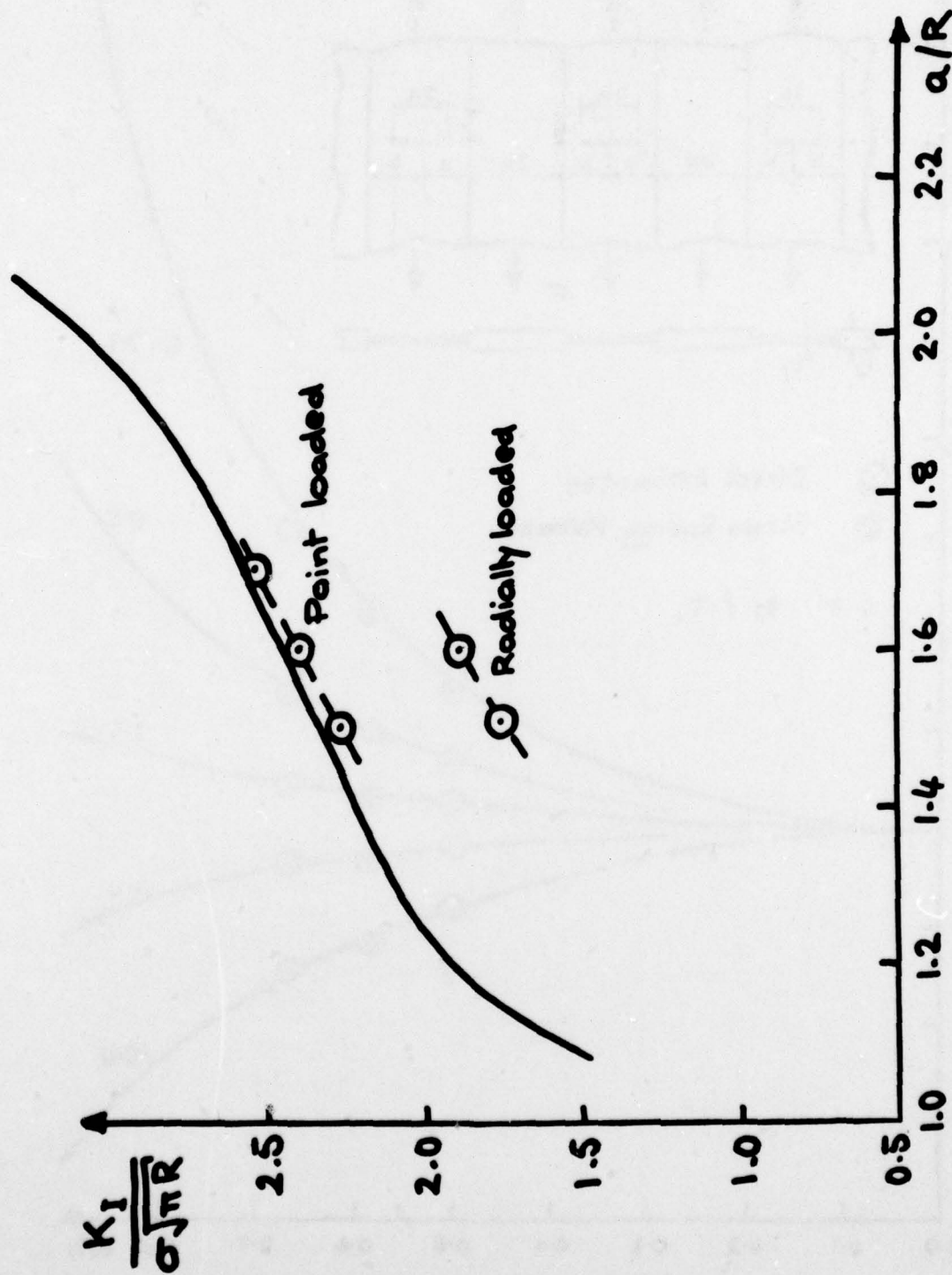Fig 5a   Effect on K_I of varying pin load and lug end geometry

Fig 5b K_I for symmetric cracks in lug (Kirkby and Rooke)
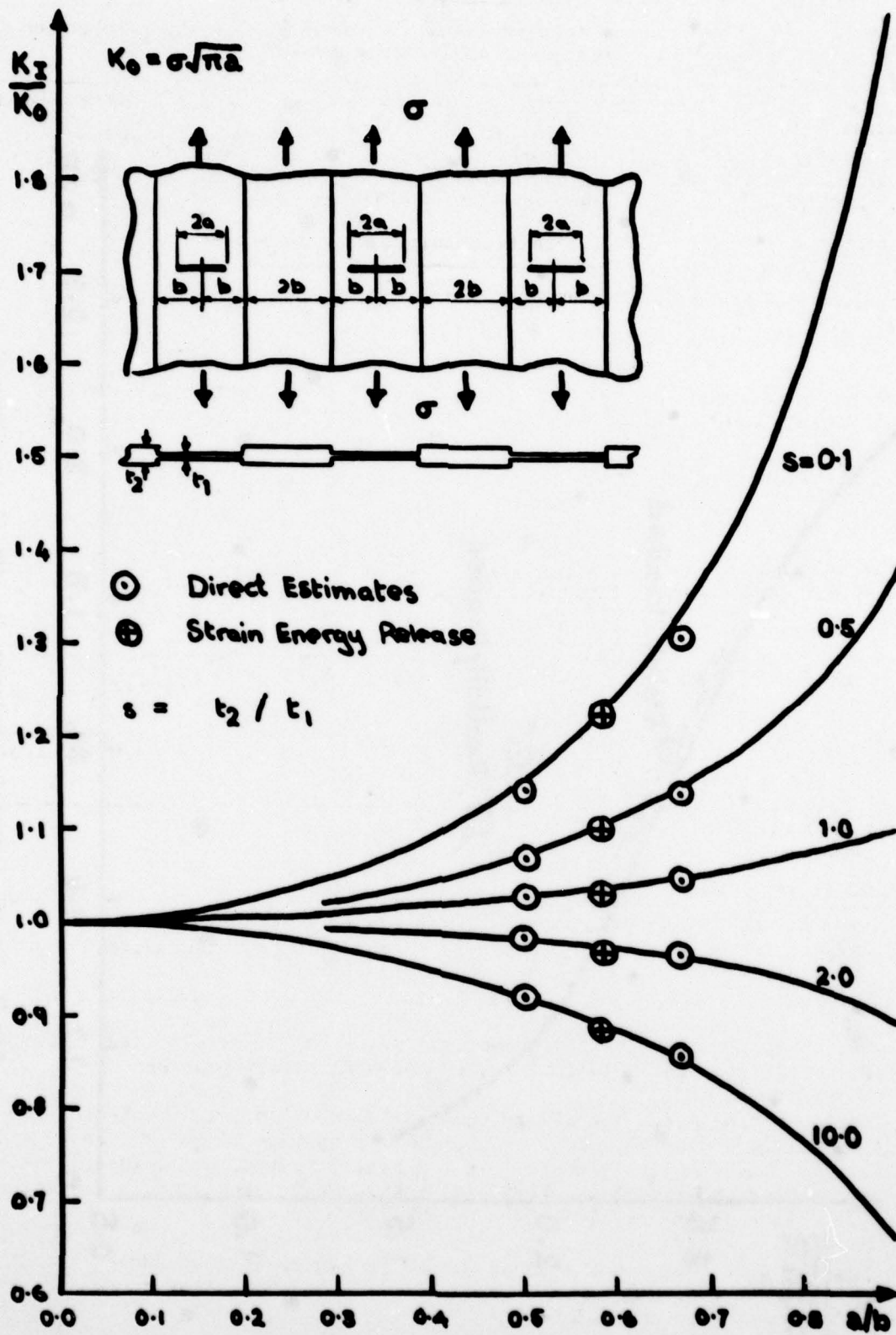
Fig 6

Fig 6 K_I for collinear cracks in periodically stiffened sheet

REPORT DOCUMENTATION PAGE

Overall security classification of this page

UNCLASSIFIED

As far as possible this page should contain only unclassified information. If it is necessary to enter classified information, the box above must be marked to indicate the classification, e.g. Restricted, Confidential or Secret.

1. DRIC Reference (to be added by DRIC)	2. Originator's Reference RAE TM Structures 940	3. Agency Reference N/A	4. Report Security Classification/Marking UNCLASSIFIED

5. DRIC Code for Originator 850100	6. Originator (Corporate Author) Name and Location Royal Aircraft Establishment, Farnborough, Hants, UK

5a. Sponsoring Agency's Code N/A	6a. Sponsoring Agency (Contract Authority) Name and Location N/A

7. Title The implementation and practical verification of a superposition method for the solution of elastic crack problems

7a. (For Translations) Title in Foreign Language

7b. (For Conference Papers) Title, Place and Date of Conference
International Conference on Numerical Methods in Fracture Mechanics

Swansea, January 9-13 1978

8. Author 1. Surname, Initials Bartholomew, P.	9a. Author 2	9b. Authors 3, 4	10. Date November 1978	Pages 19	Refs. 14

11. Contract Number N/A	12. Period N/A	13. Project	14. Other Reference Nos.

15. Distribution statement
   (a) Controlled by –

   (b) Special limitations (if any) –

16. Descriptors (Keywords)   (Descriptors marked * are selected from TEST)
   Stress intensity factor. Finite elements. Linear fracture mechanics.

17. Abstract

   This Memorandum demonstrates a superposition method which provides a practical and efficient method for the solution of linear elastic crack problems.

   The method approximates the linear elasticity solution over a substructure containing the crack tip by the superposition of appropriate singular stress fields and a relatively coarse finite element mesh. A hybrid variational principle is employed to ensure displacement compatibility between this substructure and adjoining ones employing finite elements alone.

   A key feature of the method is that it may be implemented by adding routines to existing finite element packages. The particular implementation discussed here is intended for use in the aircraft industry.

RAE Form A143